Finite-Time Stabilization of Linear Systems With Unknown Control Direction via Extremum Seeking

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***Abstract*—In this article, the finite-time stabilization problem is solved for a linear time-varying system with unknown control direction by exploiting a modified version of the classical extremum-seeking algorithm. We propose to use a suitable oscillatory input to modify the system dynamics, at least in an *average* sense, so as to satisfy a differential linear matrix inequality condition, which in turn guarantees that the system's state remains inside a prescribed time-varying hyperellipsoid in the state space. The finite-time stability (FTS) of the averaged dynamics implies the FTS of the original system, as the distance between the original and the averaged dynamics can be made arbitrarily small by choosing a sufficiently high value of the dithering frequency used by the extremum-seeking algorithm. The main advantage of the proposed approach resides in its capability of dealing with systems with unknown control direction, and/or with a control direction that changes over time. Being FTS a quantitative approach, this article also gives an estimate of the necessary minimum dithering/mixing frequency provided, and the effectiveness of the proposed finite-time stabilization approach is analyzed by means of numerical examples.**

***Index Terms*—Extremum seeking (ES), finite-time stability (FTS), Lie bracket averaging.**

I. Introduction

Extremum seeking (ES) was originally introduced in [1] as a method to find the (local) extrema of an unknown function, possibly the output of a dynamical system, which depends on one or more tunable parameters. The gist of this technique is to start from a rough estimate of the optimal parameters' value and then exploit a sinusoidal perturbation to explore the unknown map around the said estimate in order to move toward a local optimum. A formal proof of the stability of ES applied to the stable nonlinear systems with an unknown output map first appeared in the literature in 2000 (see [2] and the references therein), which made the use of a combination of averaging and singular perturbation theory.

A first attempt to extend this technique to simple linear marginally stable and unstable systems can be found in [3], where a method for tracking a target emitting a signal in the absence of any position measurement was proposed for autonomous vehicles. Although [3] regarded the stability properties of the considered system as an obstacle for the optimization of the output functional, the stabilization of the system can also be considered a goal by itself; in this view, the modified ES algorithms that minimize Lyapunov-like functions have been proposed. In particular, a possible stabilizing ES technique was originally introduced in [4], where Stanković analyzed the link between the trajectories of a system excited by a periodic, zero-average perturbation and the associated Lie-bracket averaged system [5]. It can be shown that the trajectories of the original system converge uniformly to those of the averaged system as the parameter ε, linked to the frequency and the amplitude of the perturbation, tends to 0. Moreover, exploiting the notion of *semiglobal practical stability* introduced in [6], it can be shown that if the Lie-bracket averaged system is globally uniformly asymptotically stable, then the original system is practically globally uniformly asymptotically stable for a sufficiently small value of ε, i.e., its trajectories are confined in a $\mathcal{O}(\varepsilon)$ neighborhood of the origin of the state space. Based on that, Scheinker [7] and Scheinker and M. Krstić [8] analyzed the stabilizing properties of the proposed ES scheme for a variety of systems (including linear time-varying (LTV) and nonlinear and non-affine in control systems) using different dithering signals. The proposed methodology is applied to the problem of tuning the quadrupole magnets and the bouncer cavities of a particle accelerator installed at the Los Alamos Neutron Science Centre. A great advantage of this stabilization technique is that it is capable of dealing with systems whose control direction is unknown.

Inspired by these works, in this article, we try to extend these ES stabilization results to a different kind of stability property, namely the finite-time stability (FTS) of the linear dynamical systems [9].

Finite-time (FT) stabilization is a concept linked to, but independent from, Lyapunov stabilization. In particular, a system is said to be FTS with respect to a given time horizon T, an initial time instant t_0, a positive-definite symmetric matrix R, and a positive-definite symmetric matrix-valued function of time $\Gamma(t)$ defined over the time interval $[t_0\,, t_0 + T]$, if the state trajectory starting from a point inside the hyperellipsoid defined by $x_0^T R x_0 \leq 1$ stays inside the time-varying hyperellipsoid defined by $x^T(t)\Gamma(t)x(t) < 1$.

The concept of FTS, originally introduced in the control literature in the 1960s [10]–[12], has seen a renewed interest when efficient computational tools to solve algebraic and differential linear matrix inequality (DLMI) problems became available, allowing to verify "practical" FTS conditions [13], [14] for LTV systems. More recently, the FTS problem has been tackled for hybrid systems [15], [16] as well as in the stochastic framework [17]–[19]. Such increasing interest in FTS and associated input–output notion [20] comes from the possibility of effectively adopting FTS concepts to enforce specific quantitative requirements on the transient of the closed-loop response of a control system [21].

The main idea of the present article is to apply the FTS stabilization techniques available in the literature to the Lie-bracket averaged model obtained after applying the ES controller to a LTV plant. As it will be

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discussed later, this approach has the main advantage of being able of dealing with systems whose control direction is unknown and possibly time-varying, as only the product $B(t)B^T(t)$ appears in the obtained DLMI condition. Then, we exploit the uniform convergence of the trajectories of the original system to those of the averaged one to draw conclusions on the dithering frequency needed to keep the distance between the true and the averaged state trajectories below a prescribed threshold, in order to extend the FTS properties of the averaged system to the actual one.

It is worth to remark that our approach for FTS differs from what has been proposed in [22], where the problem of tracking a desired trajectory is tackled. Oliveira *et al.* [22] showed that the proposed control approach can be also effectively used to seek the extremum of a given output function. Conversely, in this article, a modified version of the ES algorithm is used to keep the state trajectory within the prescribed bounds over a chosen FT horizon.

The rest of this article is organized as follows. Section II gives an overview of the mathematical background, including the main concepts of Lie bracket averaging and FTS. Section III presents the application of the FT control techniques presented in [9] to the Lie-bracket averaged system. In section IV, some practical indications are given for the choice of the dithering frequency of the ES scheme. Finally, Section V shows some examples of the application of the proposed technique.

II. Background Overview

In this section, some preliminary concepts are introduced. In particular, in Section II-A, the notion of the Lie-bracket averaged system associated to a dynamical system subject to periodic inputs is presented; in Section II-B, the notion of systems with converging trajectories, i.e., state trajectories whose distance can be made arbitrarily small by acting on a parameter, is discussed. Finally, in Section II-C, the concept of FTS of a linear system is recalled, together with some necessary and sufficient conditions.

Notation: In the following, $||\cdot||$ denotes the norm of a matrix, whereas $|\cdot|$ denotes the norm of a vector. Moreover, given two symmetric matrices M and N, $M \prec 0$ indicates that M is negative-definite, $M \succ 0$ indicates that it is positive definite, $M \preceq 0$ indicates that it is negative-semidefinite, and $M \succeq 0$ indicates that it is positive semidefinite; $M \prec N$ is equivalent to $M - N \prec 0$ and similarly for $M \preceq N$, $M \succ N$, and $M \succeq N$. $\mathrm{Dom}(\cdot)$ indicates the domain of a function. Finally, $[f(\cdot)]_{t_0}^{t} := f(t) - f(t_0)$.

A. Lie Bracket Averaging

Consider a system in the general form as follows:

$$\dot{x}(t) = \sum_{i=1}^{m_1} b_i(x)u_i(t) + \frac{1}{\sqrt{\varepsilon}}\sum_{i=1}^{m_2} \hat{b}_i(x)\hat{u}_i(t,\theta)$$
$$x(t_0) = x_0 \tag{1}$$

where the functions $\hat{u}_i(t,\theta)$ are T_u-periodic in $\theta = t/\varepsilon$ with zero average on the period T_u. The Lie-bracket averaged system [8] associated to (1) is as follows:

$$\dot{\bar{x}}(t) = \sum_{i=1}^{m_1} b_i(\bar{x})u_i(t) + \frac{1}{T_u}\sum_{i=1,i<j}^{m_2} [\hat{b}_i,\hat{b}_j](\bar{x})\nu_{ij}(t)$$
$$\bar{x}(t_0) = x(t_0) \tag{2}$$

where $\nu_{\mathrm{ij}}(t)$ is defined as

$$\nu_{ij}(t) = \int_0^{T_u}\int_0^{\theta} \hat{u}_i(t,\sigma)\hat{u}_j(t,\theta)d\sigma d\theta$$

and $[\hat{b}_i,\hat{b}_j](x)$ is the standard Lie bracket of $\hat{b}_i(x), \hat{b}_j(x)$

$$[\hat{b}_i,\hat{b}_j](x) = \frac{\partial \hat{b}_j(x)}{\partial x}\hat{b}_i(x) - \frac{\partial \hat{b}_i(x)}{\partial x}\hat{b}_j(x).$$

Note that this definition holds for all the integer multiples nT_u of the period T_u.

B. Converging Trajectories Property

The basic hypothesis that underlies the method proposed in this article is the convergence of trajectories [6] for the original and the averaged systems. Consider a generic system

$$\dot{x}(t) = f(t,x) \tag{3}$$

and its perturbed counterpart

$$\dot{x}^{(\varepsilon)}(t) = f^{(\varepsilon)}(t,x^{(\varepsilon)}) \tag{4}$$

where the superscript (ε) indicates the dependence of the system dynamics on the small parameter ε [e.g., as in (1)]. Denote by $\Phi(t,t_0,x_0)$ and $\Phi^{\varepsilon}(t,t_0,x_0)$, the solutions of (3) and (4), respectively, passing through the point x_0 at $t = t_0$. Systems (3) and (4) are said to have converging trajectories if $\forall \hat{T} > 0 \quad \forall K$ is a compact subset of $\mathbb{R}^n$ such that $\{t \in [t_0, t_0+\hat{T}], x \in K\} \in \mathrm{Dom}(\Phi)$ and $\forall \Delta > 0$, $\exists\tilde{}\varepsilon^* > 0$ such that $\forall\tilde{}t_0 \in \mathbb{R}, \forall\tilde{}x_0 \in K$ and $\forall\tilde{}\varepsilon \in (0,\varepsilon^*)$

$$\left|\Phi^{(\varepsilon)}(t,t_0,x_0) - \Phi(t,t_0,x_0)\right| < \Delta \qquad \forall t \in [t_0, t_0+\hat{T}].$$

In particular, this property holds for a given system in the form (1) and the corresponding Lie-bracket averaged system (2) [6], [8], [23], in the sense that, given the period $T_u < 0$, $n \in \mathbb{N}$, then $\exists\,\varepsilon^* > 0: \ \forall\,\varepsilon \in (0,\varepsilon^*)$

$$\max_{t\in[0,nT_u]} |x(t) - \bar{x}(t)| \le \Delta(nT_u,\varepsilon)$$

where $\Delta \to 0$ as $\varepsilon \to 0$.

C. FT Stabilization (With Ellipsoidal Domains)

We now recall the definition of FTS [9] of a LTV system. Generally speaking, given a positive-definite, symmetric matrix R and a positive-definite, symmetric matrix-valued function $\Gamma(t)$ defined over a time interval $[t_0, t_0+T]$, and an autonomous LTV system in the form

$$\dot{x}(t) = A(t)x(t), \qquad x(t_0) = x_0 \tag{5}$$

is said to be FTS with respect to $(t_0, T, \Gamma(\cdot), R)$, iff, by definition

$$x_0^T R x_0 \le 1 \Rightarrow x^T(t)\Gamma(t)x(t) < 1, t \in [t_0, t_0+T]. \tag{6}$$

Note that, for this definition to be well-posed, it must hold true that $\Gamma(t_0) \prec R$.

In [9, Th. 2.1], several equivalent conditions are given in order to assess if a system in the form (5) is FTS. These conditions can be extended to the state feedback closed-loop system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{7a}$$
$$u = K(t)x(t). \tag{7b}$$

In particular, the system (7) is said to be FT stabilizable by a linear static state feedback control law w.r.t. $(t_0, T, \Gamma(\cdot), R)$ iff [9, Th. 3.1]

$$\begin{cases} -\dot{Q}(t) + Q(t)A^T(t) + A(t)Q(t) + L^T(t)B^T(t) \\ +B(t)L(t) \prec 0 \\ Q(t) \prec \Gamma^{-1}(t), \quad t \in [t_0, t_0+T] \\ Q(t_0) \succ R^{-1}, \quad t \in [t_0, t_0+T]. \end{cases} \tag{8}$$

If condition (8) is satisfied for some $Q(t)$ and $L(t)$, then the controller gain that FT stabilizes the system is given by

$$K(t) = L(t)Q^{-1}(t)\,. \tag{9}$$

III. FT Stabilization Via ES and Lie Bracket Averaging

We are now ready to introduce the main contribution of this article, namely the FT stabilization of the single-input LTV systems via ES.

Let us consider a single-input LTV system in the form

$$\dot{x}(t) = A(t)x + B(t)u\,, \qquad x(t_0) = x_0 \tag{10a}$$

where $B(t) \in \mathbb{R}^{n\times 1}$, and the following control law

$$u(t) = \alpha\sqrt{\omega}\cos(\omega t) - k\sqrt{\omega}\sin(\omega t)x^T\Pi(t)x \tag{10b}$$

where the $\Pi(t)$ matrix is assumed to be symmetric and positive definite. This particular choice of the control term is similar to the one proposed in [8]; the reason behind this choice is that if $\Pi(t) = \Gamma(t)$ is chosen, then it is possible to directly take into account the quantity $x^T(t)\Gamma(t)x(t)$ that appears in the FTS problem statement (6), similarly to what is done with L_gV controllers in Lyapunov techniques. Moreover, the setting $\Pi(t) = \Gamma(t)$, which is, indeed, made in Section IV, will also turn useful in deriving a practical criterion for the choice of the dithering frequency. However, in order to keep the discussion as general as possible, a generic symmetrical positive-definite matrix $\Pi(t)$ is considered until this assumption is explicitly introduced in Section IV. Now, by applying (2) to (10), provided that the quantity $1/\omega$ is small enough for the averaging results to hold, the following Lie-bracket averaged system is obtained

$$\dot{\bar{x}}(t) = A(t)\bar{x} - k\alpha B(t)B^T(t)\Pi(t)\bar{x} \tag{11}$$

where $\frac{1}{\omega}$ plays the role of the small parameter ε.

If the averaged system can be FT stabilized, then the converging trajectories property stated in Section II-B ensures that the state of the closed-loop system will be drawn toward the desired region of the phase space as $\omega \to \infty$.

Clearly, the FTS of the averaged system does not automatically imply the FTS of the closed-loop system, because the oscillations of $x(t)$ around the averaged trajectory $\bar{x}(t)$ could still violate the requirement $x^T(t)\Gamma(t)x(t) < 1$. However, thanks to the convergence of trajectories, for any given value of Δ (i.e., the maximum allowed distance between x and $\bar{x}$), it is always possible [5], [23] to find a minimum frequency ω^* such that $|x - \bar{x}| < \Delta \quad \forall \omega > \omega^*$. Thus, proper FT stabilization of system (10) can be achieved by FT-stabilizing the averaged system (11) with respect to an opportune smaller region in state space, and then by choosing a frequency ω such that the distance between the boundaries of these two regions is not exceeded by the distance of the state trajectory from its average (see Fig. 1).

This observation leads us to establish the following lemma.

Lemma 1: Consider the two hyperellipses defined by

$$x^T\Gamma x = 1\,, \quad y^T\Gamma y = r^2 \tag{12}$$

where Γ is a $n \times n$ positive-definite matrix and $x, y \in \mathbb{R}^n$. Then, the minimum distance between the two hyperellipses is given by $(1 - r)\min_i \gamma_i$, where $\{\frac{1}{\gamma_i^2}\}$ is the set containing all the eigenvalues of Γ.

Proof: First of all, assume, without loss of generality, that $\Gamma = \mathrm{diag}(\frac{1}{\gamma_1^2}, \frac{1}{\gamma_2^2}, \ldots, \frac{1}{\gamma_n^2})$ Indeed, since Γ is positive definite, there always exists an orthonormal matrix V such that $\Gamma V = VD$, with D diagonal. Now, consider the quantity

$$1 - r^2 = \sum_{i=1}^{n}\frac{x_i^2 - y_i^2}{\gamma_i^2} = \sum_{i=1}^{n}(x_i - y_i)\frac{x_i + y_i}{\gamma_i^2}\,.$$

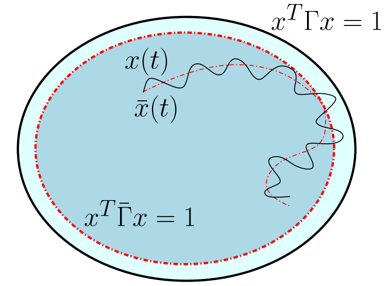

Fig. 1. Aim of the proposed technique is to FT stabilize the averaged dynamics $\bar{x}(t)$ (dash-dotted red trajectory) with respect to $\bar{\Gamma}(t)$ (the ellipse defined by $x^T\bar{\Gamma}x = 1$ is shown by the dash-dotted red line). If the distance between $x(t)$ and $\bar{x}(t)$ is bounded, the matrix $\bar{\Gamma}(t)$ can be chosen so as to guarantee that the state trajectory $x(t)$ is FTS with respect to $\Gamma(t)$ by choosing $\bar{\Gamma}(t)$ appropriately.

By applying the Cauchy–Schwarz inequality, we find

$$\begin{aligned} 1 - r^2 &\le \left[\sum_{i=1}^{n}(x_i - y_i)^2\right]^{1/2}\left[\sum_{i=1}^{n}\frac{(x_i + y_i)^2}{\gamma_i^4}\right]^{1/2} \\ &\le \left[\sum_{i=1}^{n}(x_i - y_i)^2\right]^{1/2}\left[\sum_{i=1}^{n}\frac{(x_i + y_i)^2}{\gamma_i^2}\right]^{1/2}\frac{1}{\min_i(\gamma_i)} \\ &= \frac{|x - y|}{\min_i(\gamma_i)}\left[\sum_{i=1}^{n}\frac{(x_i + y_i)^2}{\gamma_i^2}\right]^{1/2}. \end{aligned}$$

Then, by applying the triangular inequality to the term in square brackets

$$\begin{aligned} 1 - r^2 &\le \frac{|x - y|}{\min_i(\gamma_i)}\left\{\left[\sum_{i=1}^{n}\frac{x_i^2}{\gamma_i^2}\right]^{1/2} + \left[\sum_{i=1}^{n}\frac{y_i^2}{\gamma_i^2}\right]^{1/2}\right\} \\ &= \frac{|x - y|}{\min_i(\gamma_i)}(1 + r) \end{aligned}$$

hence, $|x - y| \ge (1 - r)\min_i \gamma_i$. Observe that the equality is attained when x and y are aligned with the minor semiaxes of the hyperellipsoids defined in (12). To conclude the proof, we observe that, for a generic V matrix, we can consider the distance

$$|V(x - y)| \le ||V|| \cdot |x - y| = |x - y|$$

With this choice, $(Vx)^T\Gamma(Vx) = x^TDx$ and

$$|x - y| \ge |V(x - y)| \ge (1 - r)\min_i \gamma_i\,.$$

Lemma 2: For $x(t), \bar{x}(t) \in \mathbb{R}^n$ for a given $\Delta > 0$, assume

$$|x(t) - \bar{x}(t)| < \Delta\,.$$

Consider two symmetric, positive-definite matrix valued functions of time $\Gamma(t)$ and $\bar{\Gamma}(t)$ such that $\forall\, t \in [t_0\,, t_0 + T]$

$$\bar{\Gamma}(t) = \frac{1}{r(t)^2}\Gamma(t)\,.$$

If the inequality $\bar{x}^T(t)\bar{\Gamma}(t)\bar{x}(t) < 1$ holds on the time interval $t \in [t_0, t_0 + T]$, then, on the same interval

$$x^T(t)\Gamma(t)x(t) < 1 \quad \forall\, r(t) \le 1 - \frac{\Delta}{\min_i(\gamma_i(t))}$$

where $\{\frac{1}{\gamma_i(t)^2}\}$ are the eigenvalues of $\Gamma(t)$ at time t.

Proof: The result is obtained immediately by applying Lemma 1 at each time instant t. ■

Remark 1: For well posedness, it must hold that $\Delta < \min_{i,t}\{\gamma_i(t)\} \quad \forall t \in [t_0, t_0 + T]$.

Using Lemma 2, we can now state the following result.

Theorem 1: Consider the LTV system (10) and its averaged version (11). Suppose that the dithering/mixing frequency ω is chosen so that $|x(t) - \bar{x}(t)| < \Delta$ for $t \in [t_0, t_0 + T]$.

If the following DLMI condition is satisfied for some $Q(t), k, \alpha$, and $\Pi(t)$:

$$\begin{cases} -\dot{Q}(t) + Q(t)A(t) + A^T(t)Q(t) - k\alpha Q(t)\Pi(t)B(t)B^T(t) & \\ -k\alpha B(t)B^T(t)\Pi(t)Q(t) \prec 0 & \forall t \in [t_0, t_0 + nT] \\ Q(t) \prec \bar{\Gamma}^{-1}(t) & \forall t \in [t_0, t_0 + nT] \\ Q(t_0) \succ R^{-1} & \end{cases} \tag{13}$$

where $\bar{\Gamma}(t)$ is the matrix-valued function of time defined in Lemma 2 with

$$r(t) \leq 1 - \frac{\Delta}{\min_i(\gamma_i(t))}$$

then the closed-loop system (10) is FTS with respect to $(t_0, T, R, \Gamma(t))$ for the same values of k, α, and $\Pi(t)$.

Proof: Comparing (7) to (11), we choose

$$K(t) = -k\alpha B^T(t)\Pi(t). \tag{14}$$

For Lemma 1, the FTS of the averaged system (11) with respect to $(t_0, T, R, \bar{\Gamma}(t))$ implies the FTS of the closed-loop system (10) with respect to $(t_0, T, R, \Gamma(t))$.

Condition (13) is immediately obtained by combining (8), (9), and (14) and substituting $\Gamma(t)$ with $\bar{\Gamma}(t)$ in the FTS problem formulation for the averaged system. ■

Remark 2: Note that Δ must be small enough so that the well-posedness condition $\bar{\Gamma}(t_0) \prec R$ is still satisfied. ▲

By applying Theorem 1, we obtain the DLMI problem (13), which is still nonlinear, as it contains the product of the design parameters $k, \alpha, \Pi(t)$, and $Q(t)$. To solve it, observe [24] that the term $k\alpha BB^T\Pi\bar{x}$ in (11) is proportional to the gradient of the Lyapunov-like function $V(t, x) = x^T\Pi(t)x$, evaluated for $x = \bar{x}$. This term is weighted by the positive semidefinite matrix $B(t)B(t)^T$. If the product $k\alpha$ is large enough, under a condition of persistency of excitation of $B(t)$, this gradient term dominates the $A(t)\bar{x}$ term, and the trajectory of the averaged system evolves according to a gradient descent of $V(t, \bar{x})$. According to the definition of FTS, we want to keep the quantity $x(t)^T\Gamma(t)x(t)$ below 1, therefore it makes sense to choose $\Pi(t) = \Gamma(t)$. This choice will also turn useful in the calculations of Section IV. We can, then, perform a scan in the product $k\alpha$ in order to find a solution in terms of $Q(t)$. It is worth to remark that the proposed technique gives no particular prescription on how to tune these parameters, as the averaged system dynamics only depends on their product. However, their choice can influence the stability properties of the original system, amplitude of the oscillations, and capability of the algorithm of escaping the local minima of $V(x)$. For a discussion on the choice of k and α, see [8, Section 1.3].

IV. Practical Choice of the Dithering Frequency

As mentioned in the previous sections, the original and averaged systems exhibit so-called converging trajectories. In particular, it can be shown that given a distance Δ, it is always possible to find a minimum frequency ω^* such that the distance $|x(t) - \bar{x}(t)|$ is smaller than Δ for all $\omega > \omega^*$. This means that, once the dithering frequency has been chosen such that the condition $\omega > \omega^*$ is satisfied and the control matrix-valued function $\Pi(t)$ has been fixed, Theorem 1 can be applied to find the values of the design parameters k and α that guarantee the FTS of a system in the form (10) by means of the equivalent FTS problem formulated in terms of its autonomous Lie-bracket averaged counterpart (11). We now turn our attention to the problem of finding an estimate of the minimum dithering frequency needed for this modified ES algorithm.

For simplicity, we will consider the case where the $B(t)$ matrix is a constant of unknown sign, say $B(t) = B$. Moreover, let us fix $\Pi(t) = \Gamma(t)$ and assume $x(t_0) = \bar{x}(t_0) = x_0$ (note that the dithering signal can always be chosen so as to be 0 at $t = t_0$).

Direct integration of (10) gives

$$x(t) = x_0 + \int_{t_0}^{t} A(\tau)x(\tau)d\tau + \frac{\alpha}{\sqrt{\omega}}B\left[\sin(\omega\tau)\right]_{t_0}^{t} - \int_{t_0}^{t} Bk\sqrt{\omega}\sin(\omega\tau)\left(x^T(\tau)\Gamma(\tau)x(\tau)\right)d\tau.$$

Integrating by parts the last term, using again (10), the fact that $x^T(t)\Gamma(t)B = B^T\Gamma(t)x(t)$ (it is scalar) and applying the standard trigonometric identities we have (time dependencies are dropped for clarity)

$$\begin{aligned} x(t) = x_0 &+ \int_{t_0}^{t}\left[A - k\alpha BB^T\Gamma\right]xd\tau \\ &+ \frac{\alpha}{\sqrt{\omega}}B\left[\sin(\omega\tau)\right]_{t_0}^{t} + \frac{k}{\sqrt{\omega}}B\left[\cos(\omega\tau)x^T\Gamma x\right]_{t_0}^{t} \\ &- \int_{t_0}^{t}\frac{k}{\sqrt{\omega}}B\cos(\omega\tau)\left(x^T\dot{\Gamma}x\right)d\tau \\ &- \int_{t_0}^{t}\frac{2\tilde{}k}{\sqrt{\omega}}B\cos(\omega\tau)\left(x^T\Gamma Ax\right)d\tau \\ &- \int_{t_0}^{t}k\alpha\cos(2\omega\tau)BB^T\Gamma xd\tau \\ &- \int_{t_0}^{t}k^2\sin(2\omega\tau)BB^T\Gamma x\left(x^T\Gamma x\right)d\tau. \end{aligned} \tag{15}$$

This expression is *exact*. In particular, one possibility to find a lower bound on the dithering frequency ω would be to integrate by parts the terms depending on $2\omega\tau$ that appear on the last rows of (15)

$$\begin{aligned} &\int_{t_0}^{t}k\alpha\cos(2\omega\tau)BB^T\Gamma xd\tau \\ &\quad= \left[\frac{k\alpha}{2\omega}BB^T\Gamma x\sin(2\omega\tau)\right]_{t_0}^{t} \\ &\quad- \int_{t_0}^{t}\frac{k\alpha}{2\omega}BB^T\sin(2\omega\tau)\left[\dot{\Gamma}x + \Gamma\dot{x}\right]d\tau \end{aligned}$$

$$\begin{aligned} &\int_{t_0}^{t}k^2\sin(2\omega\tau)BB^T\Gamma x\left(x^T\Gamma x\right)d\tau \\ &\quad= \left[-\frac{k^2}{2\omega}BB^T\Gamma x\left(x^T\Gamma x\right)\cos(2\omega\tau)\right]_{t_0}^{t} \\ &\quad+ \int_{t_0}^{t}\frac{k^2}{2\omega}BB^T\cos(2\omega\tau)\left(\dot{\Gamma}x + \Gamma\dot{x}\right)\left(x^T\Gamma x\right)d\tau \\ &\quad+ \int_{t_0}^{t}\frac{k^2}{2\omega}BB^T\cos(2\omega\tau)\Gamma x\left(x^T\dot{\Gamma}x + 2x^T\Gamma\dot{x}\right)d\tau \end{aligned}$$

to obtain, along the lines of [23, Th. 1], an expression in the form

$$x - \bar{x} = \int_{t_0}^{t} \left[A - k\alpha BB^T\Gamma\right](x - \bar{x})d\tau + \sum_i R_i$$

where each remainder term R_i satisfies $|R_i| \le \frac{c_i}{\sqrt{\omega}}$, for some constant c_i independent of t_0 and x_0 and for ω large enough, under some (reasonable) assumptions. Then, the Gronwall–Bellman lemma can be applied to obtain an upper bound on the distance between the actual and averaged trajectories, which can be made arbitrarily small by increasing ω. However, the need for several partial integrations leads to cumbersome calculations, and to a result which is not readily interpretable. Moreover, the exploitation of the Gronwall–Bellman lemma easily leads to very conservative estimates. Hence, we propose to exploit the intrinsic *time-scale separation* property of the algorithm in order to draw an approximate expressionfor $x(t) - \bar{x}(t)$. This leads us to invoke the following approximation.

Approximation 1: Exploit the *time-scale separation* property of ES, and assume that the oscillations of the dithering and mixing terms vary on a much faster scale than the other terms appearing in the integrals of (15).

The whole ES method is based on the implicit assumption that all the terms in the right-hand side of (15) but the ones related to the average dynamics, i.e., $x_0 + \int_{t_0}^{t}[A - k\alpha BB^T\Gamma]xd\tau$, vanish for $\omega \to \infty$. Hence, a "safe" approximation is to assume everywhere that, for a generic function of time $f(t)$

$$\int_{t_0}^{t} f(\tau)\sin(\omega\tau)d\tau = \left[\frac{f(\tau)}{\omega}\cos(\omega\tau)\right]_{t_0}^{t} + \int_{t_0}^{t}\frac{\dot{f}(\tau)}{\omega}\cos(\omega\tau)d\tau \cong \left[\frac{f(\tau)}{\omega}\cos(\omega\tau)\right]_{t_0}^{t}$$

i.e., $\dot{f}(t) << \omega$. This leads to Approximation (15) as

$$\begin{aligned} x(t) \cong\ & x_0 + \int_{t_0}^{t}\left[A - k\alpha BB^T\Gamma\right]xd\tau \\ & + \frac{\alpha}{\sqrt{\omega}}B\left[\sin(\omega\tau)\right]_{t_0}^{t} + \frac{k}{\sqrt{\omega}}B\left[\cos(\omega\tau)x^T\Gamma x\right]_{t_0}^{t} \\ & - \frac{k}{\omega\sqrt{\omega}}B\left[x^T\dot{\Gamma}x\sin(\omega\tau)\right]_{t_0}^{t} \\ & + \frac{2\,k}{\omega\sqrt{\omega}}B\left[x^T\Gamma Ax\sin(\omega\tau)\right]_{t_0}^{t} \\ & - \frac{k\alpha}{2\omega}BB^T\left[\Gamma x\sin(2\omega\tau)\right]_{t_0}^{t} \\ & + \frac{k^2}{2\omega}BB^T\left[\Gamma x\left(x^T\Gamma x\right)\cos(2\omega\tau)\right]_{t_0}^{t}. \end{aligned} \tag{16}$$

Approximation 2: Since we are looking for a relatively large dithering frequency, we neglect the highest order terms in $1/\sqrt{\omega}$ (i.e., those with $\omega\sqrt{\omega}$ at the denominator).

This leads to

$$\begin{aligned} x(t) \cong\ & x_0 + \int_{t_0}^{t}\left[A - k\alpha BB^T\Gamma\right]xd\tau \\ & + \frac{\alpha}{\sqrt{\omega}}B\left[\sin(\omega\tau)\right]_{t_0}^{t} + \frac{k}{\sqrt{\omega}}B\left[\cos(\omega\tau)x^T\Gamma x\right]_{t_0}^{t} \\ & - \frac{k\alpha}{2\omega}BB^T\left[\Gamma x\sin(2\omega\tau)\right]_{t_0}^{t} \\ & + \frac{k^2}{2\omega}BB^T\left[\Gamma x\left(x^T\Gamma x\right)\cos(2\omega\tau)\right]_{t_0}^{t}. \end{aligned} \tag{17}$$

Approximation 3:
Assume $x^T\Gamma(t)x < 1$.

If ω is large enough, $|x(t) - \bar{x}(t)| < \Delta$ and the assumptions of Theorem 1 are satisfied. In turn, this implies that the FTS condition is satisfied for the controlled system, and thus approximation 3 holds (see also the similar argument used in [24]).

Remark 3: Intuitively, if $|B(t)|, k,$ and α are of order ≈ 1 or below, approximations 2 and 3 reduce to $\omega^{3/2} >> ||\dot{\Gamma}(t)||, ||\Gamma(t)A(t)||$, i.e., the dithering frequency needs to be much faster than the variations of the quantity $x^T(t)\Gamma(t)x(t)$. ▲

This allows to obtain the following (approximate) inequality

$$x(t) \le x_0 + \int_{t_0}^{t}\left[A - k\alpha BB^T\Gamma\right]xd\tau + \frac{2(\alpha+k)}{\sqrt{\omega}}|B| + \frac{k(\alpha+k)}{\omega}|B|^2\max_t\left\{\frac{\bar{\sigma}(\Gamma(t))}{\sqrt{\underline{\sigma}(\Gamma(t))}}\right\} \tag{18}$$

where $\bar{\sigma}(\Gamma(t))$ is the maximum eigenvalue of $\Gamma(t)$, $\underline{\sigma}(\Gamma(t))$ is its minimum eigenvalue, and we used the fact (given without demonstration) that

$$x^T\Gamma x < 1 \Rightarrow |\Gamma x| \le \frac{\bar{\sigma}(\Gamma(t))}{\sqrt{\underline{\sigma}(\Gamma(t))}}.$$

Let us define $\kappa = \max_t\{\frac{\bar{\sigma}(\Gamma(t))}{\sqrt{\underline{\sigma}(\Gamma(t))}}\}$ for brevity (note that κ is a measure of the hyperellipsoid elongation). By applying the Gronwall–Bellman lemma, we obtain

$$|x(t) - \bar{x}| \le \left\{\frac{2(\alpha+k)}{\sqrt{\omega}}|B| + \frac{k(\alpha+k)}{\omega}|B|^2\kappa\right\}\eta \tag{19}$$

where we defined $\eta = \max_t ||e^{\int_{t_0}^{t}[A(\tau) - k\alpha BB^T\Gamma(\tau)]d\tau}||$.

Then, from the desired condition $|x(t) - \bar{x}| \le \Delta$, we get the following inequality in terms of $1/\sqrt{\omega}$

$$\frac{2}{\sqrt{\omega}} + \frac{k\kappa|B|}{\omega} \le \frac{\Delta}{(\alpha+k)\eta|B|}. \tag{20}$$

Solving (20) for ω provides an indication of the minimum dithering frequency needed by the algorithm.

Remark 4: In Approximation 2, we neglected the terms proportional to $\frac{1}{\omega\sqrt{\omega}}$. If ω is large enough so that also the terms $\propto \frac{1}{\omega}$ are negligible in (17), expression (20) admits a neat interpretation. Indeed, it can be rewritten as

$$\omega \ge \left[\frac{2(\alpha+k)|B|\eta}{\Delta}\right]^2 \tag{21}$$

i.e., the square root of the minimum dithering frequency is inversely proportional to the required maximum distance, and is directly proportional to the terms that influence the amplitude of the perturbation injected into the system $(\alpha, k, |B|)$. Moreover, a larger ω is needed if the system exhibits growing modes, which tends to amplify an initial perturbation, whose behavior is concisely captured by η.

V. Examples

In this section, we consider two numerical examples to show the effectiveness of the proposed approach for FT stabilization via ES.

Example 1: Let us consider the following second-order LTI system:

$$\begin{aligned} \dot{x}(t) &= Ax + Bu \\ &= \begin{bmatrix} 0 & 0.01 \\ -0.1 & 0.15 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \end{aligned}$$

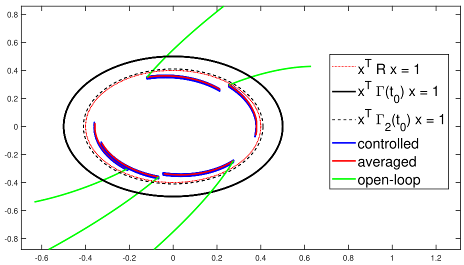

Fig. 2. State trajectory of the open-loop, controlled, and averaged system considered in Example 1 when $x_0 = [0.25 \quad 0.25]^T$. It can be seen that the open-loop trajectory (green line) is not FTS wrt the chosen Γ, R, T, and t_0. The solid black and dashed black traces represent the ellipses associated to Γ and $\tilde{\Gamma}$, while the dotted red circle represent the ellipse defined by R.

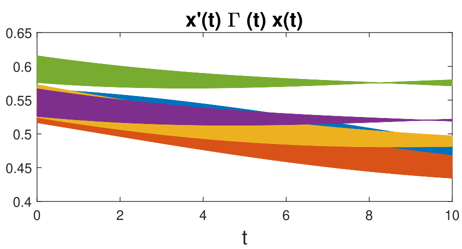

Fig. 3. Product $x(t)^T\Gamma(t)x(t)$ for different choices of the initial state for the system considered in Example 1.

where the input $u(t)$ is chosen as in (10b). We search for the values of the control parameters $k, \alpha,$ and ω, which make (1) FTS with respect to

$$R = \mathbf{I_2}/0.4\,, \;\; \Gamma(t) = \Gamma = \mathbf{I_2}/0.5$$
$$t_0 = 0\,\mathrm{s}\,, \qquad T = 10\,\mathrm{s}$$

where $\mathbf{I_n}$ is the identity matrix of order n. The maximum allowed distance between $x(t)$ and $\bar{x}(t)$ has been set to $\Delta = 0.09$ and we have chosen $\Pi = \Gamma$.

The associated DLMI problem (8) has been discretized with a time step of $T_s = 0.1$ s, with $Q(t)$ assumed to be piecewise linear, solved in MATLAB using the YALMIP [25] parser and the MOSEK [26] solver. To solve the problem, which is nonlinear, a scan of the product $k\alpha$ was performed (starting at $k\alpha = 0$ with a step of 0.01) to find the minimum value of $k\alpha$, which makes the problem feasible in $Q(t)$. The variables k and α were assumed to be constant and equal.

For this problem, we obtained the solution $k\alpha = 0.04$, with the resulting $\omega_{\min} \cong 750$ rad/s obtained from (20). For comparison, condition (21) gives a very similar value of $\omega_{\min} \cong 739$ rad/s. The fact that the first- and second-order approximated conditions (20) and (21) yield a very similar value for $\omega_{\min}$ suggests that the error introduced by Approximations 1 and 2 is negligible (note that, in this case, $\dot{\Gamma}(t) = 0$).

Figs. 2 and 3 show the obtained results for $k = \alpha, \omega = \omega_{\min}$ and five different random choices of the initial state, all such that $x_0^T R x_0 > 0.8$. In all the considered cases, the distance between the closed loop and the averaged dynamics is well below the chosen threshold Δ.

Finally, it is worth remarking again here that this approach still works even when the control direction is reversed, making it appealing for systems with unknown control direction.

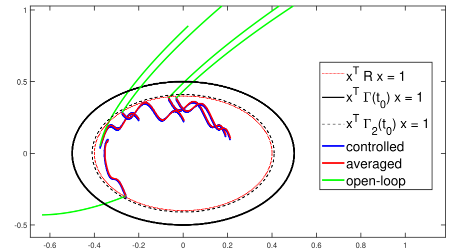

Fig. 4. State trajectory of the open-loop, controlled, and averaged system considered in Example 2. It can be seen that the open-loop trajectory (green line) is not FTS wrt the chosen Γ, R, T, and t_0. The solid black and dashed black traces represent the ellipses associated to Γ and $\tilde{\Gamma}$, while the dotted red circle represents the ellipse defined by R.

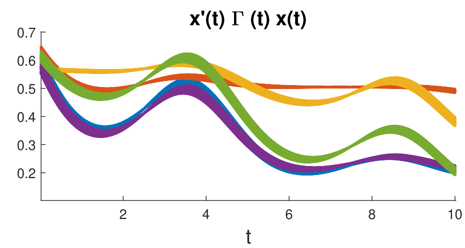

Fig. 5. Product $x(t)^T\Gamma(t)x(t)$ for five random choices of the initial state of the system considered in Example 2.

Example 2: In this example, we consider again the FTS problem of Example 1, but this time the B matrix is given by

$$B(t) = \begin{bmatrix} 0 \\ \cos(\frac{2\pi}{T}t) \end{bmatrix}.$$

The B matrix is time-varying, with a loss of controllability at $t = 2.5$ s and at $t = 7.5$ s, where $B(t) = [0 \quad 0]^T$. Problem (13) was solved using the MOSEK [26] solver discretizing the DLMI condition with a sampling time $T_s = 0.01$ s. The problem admits a solution for $k\alpha = 0.11$.

Although an explicit bound in the case of time-varying $B(t)$ was not derived in Section IV, if $B(t)$ varies on time scales, which are slower than those of the dithering/mixing signals, if Approximation 1 holds for $|\dot{B}(t)|$, we expect (20) to still provide a good approximation for $\omega_{\min}$. For this example, the value obtained by (20) is $\omega_{\min} \cong 1931$ rad/s, and again (21) provides a very close value of about 1902 rad/s. Figs. 4 and 5 show the obtained results for $k = \alpha, \omega = \omega_{\min}$ for one random choice of the initial state.

Example 3: Consider the LTV system

$$\dot{x}(t) = A(t)x + Bu$$
$$= (1 + t/10)\begin{bmatrix} 0.5 & -0.1 \\ 0 & -0.15 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

where the input $u(t)$ is chosen as in (10b). We search for the values of the control parameters $k, \alpha,$ and ω that FT stabilize (1) with respect to

$$R = \begin{bmatrix} 6.25 & 0 \\ 0 & 9.375 \end{bmatrix}, \;\; \Gamma(t) = \Gamma_0 \left(e^{t/10}\right) \text{ with } \Gamma_0 = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}$$
$$t_0 = 0\,\mathrm{s}\,, \qquad T = 5\,\mathrm{s}\,.$$

The maximum allowed distance between $x(t)$ and $\bar{x}(t)$ has been set to $\Delta = 0.0735$, and we have chosen $\Pi(t) = \Gamma(t)$.

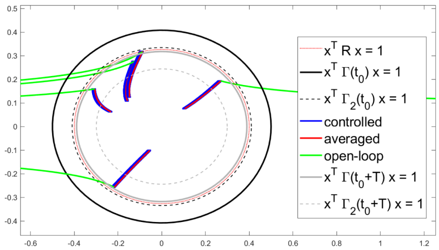

Fig. 6. State trajectory of the open-loop, controlled, and averaged system considered in Example 3. It can be seen that the open-loop trajectory (green line) is not FTS wrt the chosen Γ, R, T, and t_0. The solid black and dashed black traces represent the ellipses associated to $\Gamma(t_0)$ and $\tilde{\Gamma}(t_0)$, and the gray ones represent the ellipses defined by $\Gamma(t_0+T)$ and $\tilde{\Gamma}(t_0+T)$, while the dotted red circle represent the ellipse defined by R.

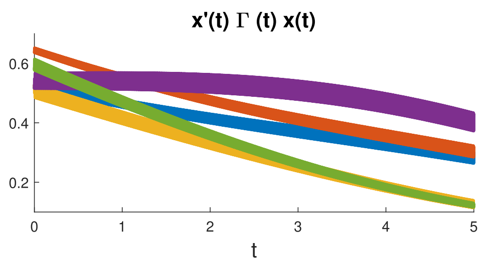

Fig. 7. Product $x(t)^T\Gamma(t)x(t)$ for five random choices of the initial state of the system considered in Example 3.

To solve the resulting DLMI, the time interval $[t_0, t_0+T]$ was discretized in 300 subintervals; $Q(t)$ was again assumed to be piecewise linear. For this problem, we obtained the solution $k\alpha = 0.14$, $\omega_{\min} \cong 1714$ rad/s obtained from (20). Condition (21) yields a very similar value of $\omega_{\min} \cong 1656$ rad/s. It can be verified that Approximations 1 and 2 are well satisfied for these values of ω.

A. Accuracy of the Proposed Bound on ω

The bound (21) is actually a conservative condition. In order to numerically assess *how* conservative this condition is, the following analysis has been carried out.

1) For Problems 1 and 3, we define the maximum allowed Δ as $\Delta_M = ||\Gamma(t_0)||^{-1/2} - ||R||^{-1/2}$ (see also Remark 1). A scan with different values of Δ in the interval $\Delta = [0.5\,, 0.95]\Delta_M$ was performed, by choosing an appropriate $\bar{\Gamma}$ for each value of Δ, and setting $k\alpha$ equal to 0.04 and 0.15, respectively, in order to stabilize all the considered cases. Values of Δ below $0.5\Delta_M$ were discarded, because a very small Δ may result into a very large ω, which due to the intrinsic double time-scale of the algorithm, may lead to very slow simulations and to numerical problems.
2) Condition (21) has, then, been used to find an estimate of the minimum frequency ω; this solution was also compared with the second-order approximation provided by (20).
3) The behavior of the closed-loop system was then simulated over an evenly spaced and slightly larger range of frequencies wrt the ones obtained from (20) and (21), and the maximum obtained distance between the actual and average trajectories for a random initial condition has been compared with the value of Δ used in (20) and (21).

The results are shown in Fig. 8. It can be seen how, fixing ω and solving for Δ, the actual maximum distance $|x(t) - \bar{x}(t)|$ is smaller then Δ by a factor $\sim 4 \div 10$ in all the considered cases. Moreover,

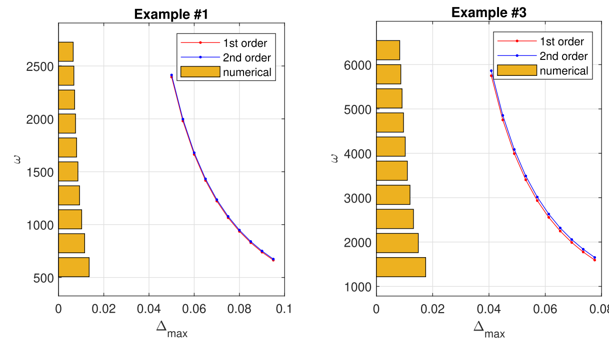

Fig. 8. Comparison of the solutions of (20) and (21) with the actual maximum distance obtained in simulation. See the main text for details.

it can be observed how the first- and second-order solutions are very close to each other.

VI. Conclusion

In this article, an approach for the FT stabilization of LTV systems with unknown control direction based on a modified version of the standard ES algorithm has been presented. The proposed methodology allows to design a static state feedback law that FT stabilizes the system in an average sense. This, in turn, implies the FT stability of the system's state trajectories under the assumption that the dithering/mixing frequency ω is chosen high enough and that the $\Gamma(t)$ matrix in the FTS definition is modified opportunely ($\tilde{\Gamma}(t)$). Approximate indications on the choice of a minimum dithering/mixing frequency are also given, taking advantage of the *time-scale separation* property on which the ES algorithm is based to derive a lower bound on $\sqrt{\omega}$ in the form of simple first- or second-order inequalities. Albeit approximate, the proposed numerical examples show that this bound is indeed capable of providing a satisfactory, and sometimes even quite conservative estimate of the minimum frequency needed, which still holds when the B matrix is slowly varying over time.

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